TUB SONLARDA GARMONIK QATOR XARAKTERISTIKASI

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ANNOTATSIYA

Ma'lumki matematikaning turli yo'nalishlarida sonli qatorlar keng qo'llaniladi. Ammo qatorlar tarkibiga kiradigan bazi turlarini yaqinlashuvchi yoki uzoqlashuvchi ekanini odatiy uslublarda ko'rsatish birmuncha qiyin yoki deyarli imkonsiz.Shu sababli tub son orqali shakllantirilgan garmonik qatorni uzoqlashuvchi ekanini 3 xil ajoyib isbotini ushbu maqolada keltirib o'tamiz.

Kalit so'zlar: *sonli qatorlar, tub sonlar, cheksiz ko'paytmalar, yaqinlashuvchi va uzoqlashuvchilik.*

CHARACTERISTIC OF HARMONIC SERIES IN PRIME NUMBERS

ABSTRACT

It is known that numerical series are widely used in various areas of mathematics. However, it is somewhat difficult or almost impossible to show that some types of series are converging or receding in conventional methods. Therefore, we have 3 wonderful proofs that the harmonic series formed by prime numbers is receding. we will quote in this article.

Keywords: *number series, prime numbers, infinite products, converging and diverging.*

KIRISH

Ushbu maqolada barcha o'rganib qolgan odatiy isbotlash usullaridan uzoqlashgan holatda masalaning isbotiga boshqa tomonlardan e'tibor bergan holda 3 xil isbotini ko'rib chiqamiz.Ma'lumki matematikada nostandart fikrlash bizni bazi masalalarga bo'lgan qarashlarimizni o'zgartiradi va natijada aniq bir masalaga boshqa sohani muhim elementlarini qo'llash imkoniyatlarini ochadi.Shu sababli biz isbot jarayonida tub sonlar bilan bog'liq ravishta sonlar nazariyasining muhim elementlarini ajoyib qo'llash jarayonlariga guvoh bo'lamiz.

Demak eslatib o'taylikki biz asosan

$$\sum_{p}\frac{1}{p}=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots\ldots+\ldots\ \ (p-tub\ son)$$

ko'rinishdagi qatorni uzoqlashuvchi ekanini isbotlashga e'tibor qaratamiz.

Hozir biz yuqorida keltirib o‘tilgan qatorni uzoqlashuvchi ekanini birinchi isbotiga to‘xtalamiz.

1-isbot: Faraz qilaylik , $\sum_p \frac{1}{p}$ qator yaqinlashuvchi bo‘lsin.U holda shunday $n \in N$ son mavjudki bunda

$$\frac{1}{p_{k+1}} + \frac{1}{p_{k+2}} + \frac{1}{p_{k+3}} + \cdots \ldots < \frac{1}{2}$$

tengsizlik bajariladi.

Buni esa quyidagicha ko‘rinishda yozib olishimiz mumkin, ya’ni $\sum_{k=n+1}^{+\infty} \frac{1}{p_k} < \frac{1}{2}$.

Avval biz $Q_m = 1 + mN \quad m = 1,2,3,,,,$ va $N = p_1 p_2 p_3 p_4 \ldots p_n$ ko‘rinishidagi Q_m sonlarni qaraylik. Ko‘rinib turibdiki Q_m sonlar $p_1, p_2, p_3, p_4 \ldots, p_n$ larning hech biriga bo‘linmaydi.Natijada kelib chiqadiki $1 + mN$ ko‘rinishidagi sonlarni tub bo‘luvchilari $\{p_{n+1}, p_{n+2},,,,,\}$ to‘plamga tegishli.

Endi quyidagi cheksiz yig‘indini qarasak

$$\left(\frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \frac{1}{p_{n+3}} + \cdots \ldots\right) + \left(\frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \frac{1}{p_{n+3}} + \cdots \ldots\right)^2 + \cdots \ldots$$
$$+ \left(\frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \frac{1}{p_{n+3}} + \cdots \ldots\right)^t + \cdots \ldots$$

Agar berilgan yig‘indi yoyilma shaklida yoziladigan bo‘lsa unda $\frac{1}{p_{n+1}^{m_1} p_{n+2}^{m_2} \ldots\ldots p_{n+\delta}^{m_\delta}}$ kabi hadlarni uchratish mumkin shuningdek ular cheksiz ko‘p.

Demak quyidagi natijani yoza olamiz

$$\frac{1}{Q_1} + \frac{1}{Q_2} + \cdots \ldots + \frac{1}{Q_m} \le \sum_{t=1}^{+\infty} \Big(\sum_{k=n+1}^{+\infty} \frac{1}{p_k} \Big)^t \quad <=> \quad \sum_{k=1}^{m} \frac{1}{Q_k} \le \sum_{t=1}^{+\infty} \Big(\sum_{k=n+1}^{+\infty} \frac{1}{p_k} \Big)^t \qquad (1)$$

Boshqa tomondan esa

$$\sum_{k=n+1}^{+\infty} \frac{1}{p_k} < \frac{1}{2} \qquad (2)$$

Yuqorida berilgan (1) va (2) tengsizliklardan quyidagi natijani yoza olamiz.

$$\sum_{k=1}^{m} \frac{1}{Q_k} < \sum_{t=1}^{+\infty} \Big(\frac{1}{2}\Big)^t$$

Ma’lumki $\sum_{t=1}^{+\infty} (\frac{1}{2})^t$ qator yaqinlashuvchi. Demak, $\sum_{k=1}^{m} \frac{1}{Q_k}$ qator chegaralanganligini xulosa qilishimiz mumkin. Ammo shu yerda ziddiyatga kelib qoldik chunki

$$\frac{1}{Q_m} = \frac{1}{1 + mN} > \frac{1}{N + mN} = \frac{1}{N(1 + m)}$$

bu orqali aytishimiz mumkinki

$$\sum_{m=1}^{+\infty}\frac{1}{N(1+m)}=\frac{1}{N}\sum_{m=1}^{+\infty}\frac{1}{(1+m)}$$

qator uzoqlashuvchi. Demak farazimiz ziddir. Ya'ni, $\sum_p \frac{1}{p}$ qator uzoqlashadi. Endi berilgan qator uzoqlashuvchiligining ikkinchi isbotini ko'rib chiqamiz.

2-isbot: Faraz qilaylik, $\sum_p \frac{1}{p}$ qator biror haqiqiy songa yaqinlashadigan bo'lsin. U holda, shunday $k\epsilon N$ mavjudki bunda

$$\frac{1}{p_{k+1}}+\frac{1}{p_{k+2}}+\frac{1}{p_{k+3}}+\cdots\ldots<\frac{1}{2}$$

tengsizlik bajariladi. Shu tariqa biz quyidagi tengsizlikni ham yoza olamiz.

$$\frac{x}{p_{k+1}}+\frac{x}{p_{k+2}}+\frac{x}{p_{k+3}}+\cdots\ldots<\frac{x}{2}$$

bu yerda $x\epsilon N$.

Endi $N(x,p_k)$ orqali barcha musbat butun n larni sonini belgilaylik, bunda $n\leq x$ va bundan tashqari n, $p>p_k$ larda birorta tub songa bo'linadigan emas. Biz shunday $n\epsilon N$ sonni quyidagicha formada yozib olamiz

$$n=m_1^2 m$$

bu yerda

$$m=2^{b_1}*3^{b_2}*5^{b_3}*\ldots..*p_k^{b_k}\quad b_k\in\{0,1\}, i=1,2,\ldots,k$$

Ma'lumki, $b_k\in\{0,1\}$ ekanini inobatga olsak m uchun 2^k ta turli qiymatlar mavjud.

Endi $n=m_1^2 m \;=>\; \sqrt{n}=m_1*\sqrt{m} \;=>\; m_1\leq\sqrt{n}\leq\sqrt{x}$ qo'sh tengsizlikka ega bo'lamiz. Natijada ayta olamizki m_1 ning ko'pi bilan $\sqrt{x}$ ta turli qiymati mavjud va bundan kelib chiqadiki n ning ko'pi bilan $2^k*\sqrt{x}$ ta turli qiymati mavjud. Shu orqali, $N(x,p_k)\leq 2^k*\sqrt{x}$ ifodaga ega bo'lamiz.

Boshqa tarafdan esa $N(x,p_k)>\frac{x}{2}$ chunki p_k ni quyidagicha tanlay olamiz $p_k>\frac{x}{2}$ agar x-juft son bo'lsa, yoki $p_k>\frac{x+1}{2}$ agar x-toq son bo'lsa. Demak, $p>p_k$ da har bir p-tub son 1,2 ,3,….., p_k sonlarga bo'linmaydi. So'ngida biz quyidagi qo'sh tengsizlikka ega

$$\frac{x}{2}<N(x,p_k)\leq 2^k*\sqrt{x}$$

Bundan esa quyidagi bahoni olamiz.

$$\frac{x}{2}<2^k*\sqrt{x}\quad<=>\quad x<2^{2k+2}$$

Shu tariqa xulosa qila olamizki $\sum_p \frac{1}{p}$ qator uzoqlashuvchi.

Endi biz qo'yilgan masalaning uchinchi isbotiga to'xtalib o'tamiz.

3-isbot: Aytaylik, $n > 1$ natural son bo'lsin. U holda $p_1, p_2, p_3, \dots., p_k \le n \le p_{k+1}$ tengsizlik bazi k musbat butun sonlar uchun o'rinli bo'la oladi. Ko'rinib turibdiki n ga teng yoki undan kichik har bir butun sonning tub bo'luvchilari $\{p_1, p_2, p_3, \dots., p_k\}$

to'plamga tegishli.

Natijada ana shunday har bir butun sonni quyidagicha formada tasvirlay olamiz.

$$p_1^{a_1} * p_2^{a_2} * \dots \dots p_k^{a_k}$$

bu yerda $a_i \ge 0$, $i = 1,2,3, \dots, k$

Agar biz $2^m > n$ bo'ladigan m butun sonlarni qarasak u holda shubhasiz quyidagi bajariladi ya'ni $p_i^m > n$, p_i −tub son. Natijada $p_1^m * p_2^m * \dots \dots p_k^m > n$ yoki $p_1^{a_1} * p_2^{a_2} * \dots p_i^m \dots * p_k^{a_k} > n$ tengsizliklarga ega bo'lamiz.

Bizga cheksiz kamayuvchi geometrik progressiya orqali quyidagi ma'lum.

$$\frac{1}{1 - \frac{1}{p_i}} = \sum_{j=0}^{+\infty} \frac{1}{p_i^j}$$

Bundan kelib chiqadiki

$$\frac{1}{1-\frac{1}{p_i}} > 1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots \dots + \frac{1}{p_i^m} \quad => \prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i}} > \prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots \dots + \frac{1}{p_i^m})$$

Agar $\prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots \dots + \frac{1}{p_i^m})$ ko'paytmani yoysak $\frac{1}{p_1^{a_1} * p_2^{a_2} * \dots \dots * p_k^{a_k}}$ ko'rinishdagi hadlar yig'indisiga ega bo'lamiz. Bu yerda $0 \le a_i \le m$ i=1,2,3,4,,,,,k va shuningdek bizda $p_1^{a_1} * p_2^{a_2} * \dots p_i^m \dots * p_k^{a_k} > n$ tengsizlik mavjud.Yuqoridagilarni hisobga olib biz quyidagi natijaga kelamiz.

$$\prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots \dots + \frac{1}{p_i^m}) > \sum_{l=1}^{n} \frac{1}{l} \quad => \quad \prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i}} > \sum_{l=1}^{n} \frac{1}{l} \qquad (1)$$

Bizga ma'lumki quyidagi yoyilma

$$\ln(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \cdots \dots . . +(-1)^{n+1} \frac{x^n}{n} + \cdots \dots . (|x| < 1)$$

Demak, $\ln \frac{1}{1-\frac{1}{p_i}} = -\ln(1 - \frac{1}{p_i}) = -(-\frac{1}{p_i} - \frac{1}{2p_i^2} - \frac{1}{3p_i^3} - \cdots \dots \dots \dots) = \frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{3p_i^3} + \cdots \dots \dots \dots < \frac{1}{p_i} + \frac{1}{2p_i^2} + \frac{1}{2p_i^3} + \cdots \dots \dots \dots = \frac{1}{p_i} + \frac{1}{2p_i^2} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots \dots \dots \dots) = \frac{1}{p_i} + \frac{1}{2p_i^2} (\frac{1}{1-\frac{1}{p_i}})$

Ma'lumki $p_i \ge 2 \;=>\; \frac{1}{p_i} \le \frac{1}{2} \;<=>\; -\frac{1}{p_i} \ge -\frac{1}{2} \;=>\; \frac{1}{1-\frac{1}{p_i}} \le \frac{1}{1-\frac{1}{2}}$.

Bu orqali quyidagi natijalarga kelamiz.

$$-\ln(1-\frac{1}{p_i}) < \frac{1}{p_i} + \frac{1}{2p_i^2}(\frac{1}{1-\frac{1}{p_i}}) \leq \frac{1}{p_i} + \frac{1}{2p_i^2}(\frac{1}{1-\frac{1}{2}}) = \frac{1}{p_i} + \frac{1}{p_i^2}$$

$$=> \qquad \ln\frac{1}{1-\frac{1}{p_i}} < \frac{1}{p_i} + \frac{1}{p_i^2} \qquad i = 1,2,3,4,5,,,,,,,k \qquad (2)$$

$$\sum_{i=1}^{k} ln\frac{1}{1-\frac{1}{p_i}} < \sum_{i=1}^{k}\frac{1}{p_i} + \sum_{i=1}^{k}\frac{1}{p_i^2} < \sum_{i=1}^{k}\frac{1}{p_i} + \sum_{l=1}^{+\infty}\frac{1}{l^2} \ .$$

Endi $\sum_{i=1}^{k} ln\frac{1}{1-\frac{1}{p_i}} = ln\prod_{i=1}^{k}\frac{1}{1-\frac{1}{p_i}}$ ekanini hisobga olib biz quyidagi tengsizlikni hosil qilamiz.

$$\sum_{i=1}^{k}\frac{1}{p_i} > \ln\left(\sum_{l=1}^{n}\frac{1}{l}\right) - \sum_{l=1}^{+\infty}\frac{1}{l^2}$$

Berilgan tengsizlikni har bir tomonidan mos ravishta $k \to +\infty$ va $n \to +\infty$ oladigan bo‘lsak quyidagi natijalar kelib chiqadi.

Tengsizlikni o‘ng tomoniga e’tibor qaratsak

$$\sum_{l=1}^{+\infty}\frac{1}{l} = +\infty$$

va

$$\sum_{l=1}^{+\infty}\frac{1}{l^2} < +\infty$$

ekanini aniqlaymiz.

Demak, qatorni yaqinlashishi yoki uzoqlashishini baholashga ko‘ra $\sum_{i=1}^{+\infty}\frac{1}{p_i}$ qator uzoqlashuvchiligi kelib chiqadi.

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