ENG SODDA TRIGONOMETRIK TENGLAMALARNI GRAFIK USULDA YECHISH

Nurkayev Shuhrat Jurayevich
Turin politexnika universiteti akademik litseyi oliy toifali
matematika fani o'qituvchisi.

Annotatsiya: *Ushbu maqolada trigonometric tenglamalarni grafik usulda yechish tushuntirilgan. Ananaviy birlik aylanada yechish usuli maktab kursida tushintiriladi.*

Kalit so'zlar: *Trigonometrik tenglama, yechish, usul, sonlar,javob.*

Abstract: *This article explains how to solve trigonometric equations graphically. The traditional method of solving the unit circle is explained in the school course.*

Key words: *Trigonometric equation, solution, method, numbers, answer.*

Ma'lumki, maktab kursida trigonometrik tenglamalarni yechish birlik aylanada tushuntiriladi. Trigonometrik tenglamalarni grafik usulda yechish ham sezilarli darajada foyda beradi. Grafik usulda qay tarzda yechish mumkinligi tushuntiramiz.

1. $\cos x = a$ **tenglama.**

Agar $|a| > 1$, ya'ni $a \in (-\infty; -1) \cup (1; \infty)$ bo'lsa , u holda $\cos x = a$ tenglama yechimga ega emas , chunki har qanday x uchun $|\cos x| \le 1$.

Endi $|a| \le 1$, ya'ni $a \in [-1; 1]$ bo'lsin .

Ushbu $\cos x = a$ tenglamani qanoatlantiradigan barcha x larni topaylik.

Faraz qilaylik, $x \in [0; \pi]$ bo'lsin. Ushbu $x \in [0; \pi]$ kesmada yuqorida berilgan tenglamaning bitta aniq yechimi mavjud.

$$x_1 = \arccos a.$$

$y = \cos x$ funksiya –juft funksiya va demak $[-\pi; 0]$ kesmada $\cos x = a$ tenglama yana bitta yechimga ega.

$$x_2 = -\arccos a .$$

Shunday qilib , $\cos x = a$ tenglama $[-\pi;\pi]$ kesmada ikkita yechimga ega :

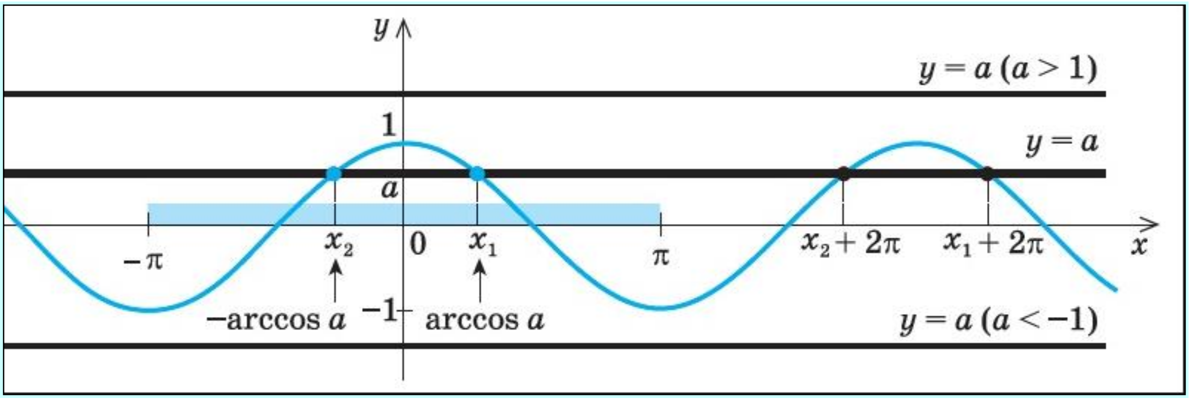
$$x = \pm\arccos a.$$

$y = \cos x$ funksiyaning davriyligi va uning davri $2\pi n$ bo‘lgani sababli qolgan yechimlarning hammasi bu yechimlardan $2\pi n, n \in Z$ ga farq qiladi .

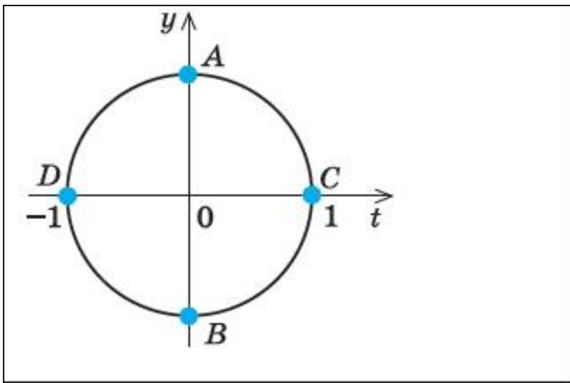
Ya’ni $\cos x = a$, $|a| \le 1$ tenglamaning barcha yechimlari quyidagicha topiladi :

$$x = \pm\arccos a + 2\pi n, n \in Z \ , \qquad \textbf{(1)}$$

$\cos x = a$ tenglamaning yechimlarini quyidagicha tasvirlash mumkin :



$\cos x = a$ tenglamaning hususiy hollardagi yechimlarini keltiramiz :



$\cos x = 0$	$x = \frac{\pi}{2} + \pi k, \ k \in Z$
$\cos x = 1$	$x = 2\pi k, \ k \in Z$
$\cos x = -1$	$x = \pi + 2\pi k, \ k \in Z$

1-misol . $\cos x = \frac{1}{2}$ **tenglamani yeching.**

Yechish:

(1) ga ko'ra :

$$x = \pm \arccos \frac{1}{2} + 2\pi n, \ n \in Z .$$

$\arccos \frac{1}{2} = \frac{\pi}{3}$ bo'lgani uchun ushbu javobga kelamiz:

$$x = \pm \frac{\pi}{3} + 2\pi n, n \in Z .$$

Javob: $x = \pm \frac{\pi}{3} + 2\pi n, n \in Z$.

2-misol . $\cos\left(2x - \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$ **tenglamani yeching.**

Yechish:

(1) ga ko'ra :

$$2x - \frac{\pi}{4} = \pm \arccos\left(-\frac{\sqrt{3}}{2}\right) + 2\pi n, n \in Z,$$

$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ bo'lgani uchun quyidagini hosil qilamiz:

$$2x = \frac{\pi}{4} \pm \frac{5\pi}{6} + 2\pi n, n \in Z,$$

yoki

$$x = \frac{\pi}{8} \pm \frac{5\pi}{12} + \pi n, n \in Z .$$

Yoki $x_1 = \frac{\pi}{8} + \frac{5\pi}{12} + \pi n = \frac{3\pi + 10\pi}{24} + \pi n = \frac{13\pi}{24} + \pi n, n \in Z$ va

$$x_2 = \frac{\pi}{8} - \frac{5\pi}{12} + \pi n = \frac{3\pi - 10\pi}{24} + \pi n = -\frac{7\pi}{24} + \pi n, n \in Z .$$

Javob: $x = \frac{\pi}{8} \pm \frac{5\pi}{12} + \pi n, n \in Z$

3-misol . $\cos^2 x = 1$ **tenglamaning nechta ildizi** $x^2 \le 10$ **shartni qanoatlantiradi ?**

Yechish:

Dastlab , $x^2 \le 10$ tengsizliklni yechib olamiz.

$$x^2 - 10 \le 0 \ .$$

Chap tomonini ko‘paytuvchilarga ajratamiz :

$$\left(x-\sqrt{10}\right)\left(x+\sqrt{10}\right) \le 0 .$$

Bu tengsizlikni oraliqlar usulida yechib, quyidagi yechimlarni hosil qilamiz:

$$x \in \left[-\sqrt{10};\sqrt{10}\right].$$

Endi berilgan trigonometrik tenglamani yechamiz.

$$\cos^2 x = 1.$$

Buning uchun daraja pasaytirish formulasidan foydalanamiz.

$$\frac{1+\cos 2x}{2} = 1$$

yoki

$$1+\cos 2x = 2$$

yoki

$$\cos 2x = 1.$$

Sodda trigonometrik tenglamani yechamiz.

$$2x = 2\pi n, n \in Z$$

yoki

$$x = \pi n, n \in Z .$$

Endi berilgan shartni qanoatlantiradigan yechimlarni topamiz.

$n=-2$ da $x=-2\pi \approx -2\cdot 3{,}14 = -6{,}28 \notin \left[-\sqrt{10};\sqrt{10}\right]$,

$n=-1$ da $x=-\pi \approx -3{,}14 \in \left[-\sqrt{10};\sqrt{10}\right]$,

$n=0$ da $x=0\in\left[-\sqrt{10};\sqrt{10}\right]$,

$n=1$ da $x=\pi\approx 3{,}14\in\left[-\sqrt{10};\sqrt{10}\right]$,

$n=2$ da $x=2\pi\approx 2\cdot 3{,}14=6{,}28\notin\left[-\sqrt{10};\sqrt{10}\right]$.

$n\le -2$ va $n\ge 2$ larda yechimlar $\left[-\sqrt{10};\sqrt{10}\right]$ kesmaga tegishli bo‘lmaydi.

Javob: 3 ta

4-misol . **Agar** $|\cos x|=2+\cos x$ **bo‘lsa ,** $2^{\cos x}+3^{\sin x}$ **ning qiymatini toping.**

Yechish:

Dastlab , son modulining ta’rifini eslatib o‘tamiz. $|a|=\begin{cases} a,\ agar\ a\ge 0\ bo'lsa \\ -a,\ agar\ a<0\ bo'lsa \end{cases}$.

Demak , $|\cos x|=\begin{cases} \cos x,\ agar\ \cos x\ge 0\ bo'lsa \\ -\cos x,\ agar\ \cos x<0\ bo'lsa \end{cases}$.

1-hol. $\cos x\ge 0$ bo‘lsin. U holda

$$\cos x=2+\cos x,$$
$$0=2 .$$

Bu holda tenglama yechimga ega emas.

2-hol . $\cos x<0$ bo‘lsin. U holda

$$-\cos x=2+\cos x$$

yoki

$$-2\cos x=2$$

yoki

$$\cos x=-1.$$

Endi $\sin x$ ning qiymatini topamiz.

$$\sin x=\pm\sqrt{1-\cos^2 x}=\pm\sqrt{1-(-1)^2}=\pm\sqrt{1-1}=0 .$$

Demak , $2^{\cos x}+3^{\sin x}=2^{-1}+3^{0}=\frac{1}{2}+1=\frac{3}{2}=1{,}5$.

Javob: 1,5.

2. $\sin x = a$ **tenglama.**

Xuddi kosinusdagi singari, agar $|a| > 1$, ya'ni $a \in (-\infty; -1) \cup (1; \infty)$ bo'lsa, $\sin x = a$ tenglama ham yechimga ega emas, chunki har qanday x uchun $|\sin x| \le 1$.

Endi $|a| \le 1$, ya'ni $a \in [-1; 1]$ bo'lsin.

Ushbu $\sin x = a$ tenglamani qanoatlantiradigan barcha x larni topaylik.

$\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ kesmada $\sin x = a$ tenglamaning bitta aniq yechimi bor:

$$x_1 = \arcsin a.$$

$\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ kesmada $y = \sin x$ funksiya kamayuvchi va -1 dan 1 gacha hamma qiymatlarni qabul qiladi.

Ildiz haqidagi teoremaga ko'ra $\sin x = a$ tenglama bu kesmada ham aniq bitta yechimga ega :

$$x_2 = \pi - \arcsin a,$$

$$\sin x_2 = \sin(\pi - x_1) = \sin x_1 = a.$$

Bundan tashqari,

$$-\frac{\pi}{2} \le x_1 \le \frac{\pi}{2}$$

bo'lgani uchun va

$$\pi - \frac{\pi}{2} \le \pi - x_1 \le \pi + \frac{\pi}{2}$$

ga egamiz, ya'ni x_2 son $\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ kesmaga tegishli.

Shunday qilib, $\sin x = a$ tenglama $\left[-\frac{\pi}{2}; \frac{3\pi}{2}\right]$ kesmada ikkita yechimga ega :

$$x_1 = \arcsin a \text{ va } x_2 = \pi - \arcsin a.$$

($a = 1$ da bu yechimlar bir xil). Sinusning davriyligini (davri $2\pi n$) hisobga olib , tenglamaning barcha yechimlarini yozish uchun quyidagi formulalarni hosil qilamiz:

$$x = \arcsin a + 2\pi n \qquad (*)$$

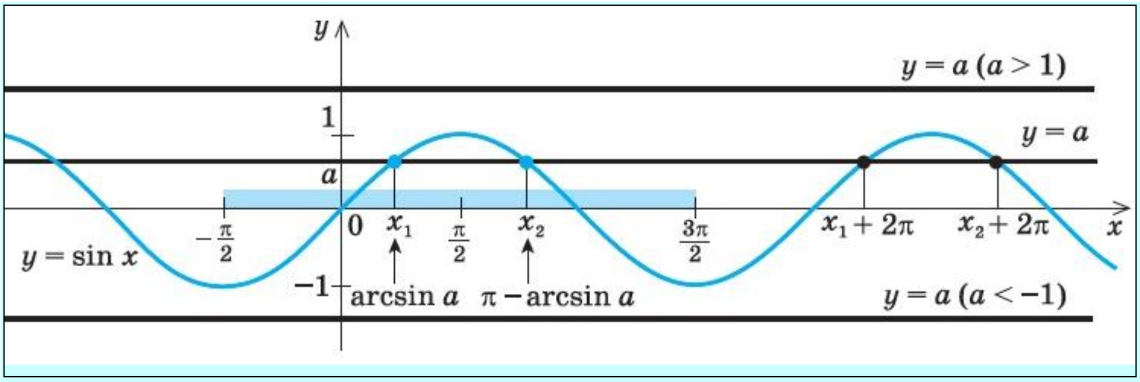
$$x = \pi - \arcsin a + 2\pi n \qquad (**)$$

Ushbu $\sin x = a$ tenglama yechimlarini ikkita emas,balki bitta formula bilan yozish qulay:

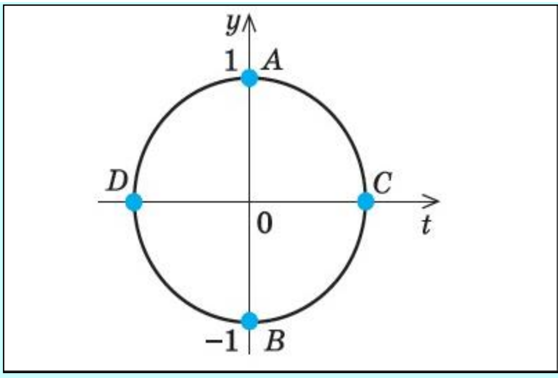
$$x = (-1)^k \cdot \arcsin a + \pi k, k \in Z \qquad \textbf{(2)}$$

$k = 2n$ larda (2) formuladan (*) formula bilan yozilgan barcha yechimlarni topishga ; $k = 2n + 1$ larda (**) formula bilan yozilgan yechimlarning barchasini topishga oson ishonch hosil qilish mumkin.

$\sin x = a$ tenglamaning yechimlarini quyidagicha tasvirlash mumkin:



$\sin x = a$ tenglamaning hususiy hollardagi yechimlarini keltiramiz:



$\sin x = 0$	$x = \pi k,\ k \in Z$
$\sin x = 1$	$x = \frac{\pi}{2} + 2\pi k,\ k \in Z$
$\sin x = -1$	$x = -\frac{\pi}{2} + 2\pi k,\ k \in Z$

5-misol. $\sin x = \frac{\sqrt{2}}{2}$ **tenglamani yeching.**

Yechish: (2) ga asosan: $x = (-1)^k \cdot \arcsin\frac{\sqrt{2}}{2} + \pi k \;, k \in Z$

Tenglamada $\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ni almashtiramiz.

$$x = (-1)^k \cdot \frac{\pi}{4} + \pi k \;, k \in Z \,.$$

Javob: $x = (-1)^k \cdot \frac{\pi}{4} + \pi k \;, k \in Z$ **.**

6-misol. $\sin x = 0,3714$ **tenglamani taqribiy yeching**.

Yechish :

(2) ga asosan

$$x = (-1)^k \cdot \arcsin 0,3714 + \pi k \;, k \in Z \,.$$

Kalkulyator yordamida topamiz:

$$\arcsin 0,3714 \approx 0,3805 .$$

Berilgan tenglamaning taqribiy yechimi quyidagicha:

$$x \approx (-1)^k \cdot 0,3805 + \pi k \;, k \in Z \,.$$

Javob: $x \approx (-1)^k \cdot 0,3805 + \pi k \;, k \in Z\,.$**.**

7-misol. $\sin\left(\frac{\pi}{10} - \frac{x}{2}\right) = \frac{\sqrt{2}}{2}$ **tenglamani yeching.**

Yechish: Sinus – toq funksiya , ya'ni $\sin(-x) = -\sin x$.

Shuning uchun berilgan tenglamani quyidagicha yozamiz:

$$\sin\left(\frac{x}{2} - \frac{\pi}{10}\right) = -\frac{\sqrt{2}}{2}.$$

(2) ga asosan quyidagini yozamiz:

$$\frac{x}{2} - \frac{\pi}{10} = (-1)^k \arcsin\left(-\frac{\sqrt{2}}{2}\right) + \pi \cdot k \;, k \in Z \,,$$

$\arcsin\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$ bo‘lgani uchun:

$$\frac{x}{2}-\frac{\pi}{10}=(-1)^k\left(-\frac{\pi}{4}\right)+\pi\cdot k\ ,k\in Z$$

yoki

$$\frac{x}{2}-\frac{\pi}{10}=(-1)^k\cdot(-1)\cdot\frac{\pi}{4}+\pi\cdot k\ ,k\in Z$$

yoki

$$\frac{x}{2}=\frac{\pi}{10}+(-1)^{k+1}\cdot\frac{\pi}{4}+\pi\cdot k\ ,k\in Z\,.$$

Tenglamaning ikkala tomonini 2 ga ko‘paytiramiz.

$$x=\frac{\pi}{5}+(-1)^{k+1}\cdot\frac{\pi}{2}+2\pi\cdot k\ ,k\in Z\,.$$

Javob: $x=\frac{\pi}{5}+(-1)^{k+1}\cdot\frac{\pi}{2}+2\pi\cdot k\ ,k\in Z\,.$

8-misol. $|\sin 3x|=\frac{1}{2}$ **tenglamani yeching.**

Yechish: $|f(x)|=a$ tenglama berilgan bo‘lsa , uni quyidagicha yozishimiz mumkin:

$$\left[\begin{array}{l} f(x)=a \\ f(x)=-a \end{array}\right.$$

Yuqoridagi trigonometrik tenglamani quyidagi ko‘rinishda yozamiz:

$$\left[\begin{array}{l} \sin 3x=\frac{1}{2} \\ \sin 3x=-\frac{1}{2} \end{array}\right.$$

(2) ga asosan quyidagilarni yozamiz:

$$\left[\begin{array}{l} 3x=(-1)^k\arcsin\frac{1}{2}+\pi k\,,k\in Z \\ 3x=(-1)^k\arcsin\left(-\frac{1}{2}\right)+\pi k\,,k\in Z \end{array}\right.$$

Ma’lumki , $\arcsin\frac{1}{2}=\frac{\pi}{6}$ va $\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$.

$$\left[\begin{array}{l} 3x=(-1)^k\dfrac{\pi}{6}+\pi k, k\in Z \\ 3x=(-1)^k\left(-\dfrac{\pi}{6}\right)+\pi k, k\in Z. \end{array}\right.$$

Tenglamalarning har birining ikkala tomonini 3 ga bo'lamiz.

$$\left[\begin{array}{l} x=(-1)^k\dfrac{\pi}{18}+\dfrac{\pi k}{3}, k\in Z \\ x=(-1)^{k+1}\dfrac{\pi}{18}+\dfrac{\pi k}{3}, k\in Z. \end{array}\right.$$

Bu ikkala yechimlarni bitta yechim ko'rinishda yozish qulay.

$$x=\pm\frac{\pi}{18}+\frac{\pi k}{3}, k\in Z.$$

Javob: $x=\pm\dfrac{\pi}{18}+\dfrac{\pi k}{3}, k\in Z$.

9-misol. $1-\sin x-\cos 2x=0$ $\left(x\in[0;2\pi]\right)$ **tenglamaning ildizlari yig'indisini toping.**

Yechish: Ma'lumki , $\sin^2 x+\cos^2 x=1$, hamda $\cos 2x=\cos^2 x-\sin^2 x$. Ushbu ayniyatlardan foydalanib, tenglamani quyidagi ko'rinishga keltiramiz:

$$\cos^2 x+\sin^2 x-\sin x-\cos^2 x+\sin^2 x=0.$$

Soddalashtirgandan so'ng quyidagi tenglikka ega bo'lamiz:

$$2\sin^2 x-\sin x=0.$$

Qavsdan tashqariga bir xil ko'paytuvchini chiqaramiz :

$$\sin x(2\sin x-1)=0.$$

Bu tenglamani yechish uchun har bir ko'paytuvchini 0 ga tenglashtirib yechamiz:

1) $\sin x=0$, $x_1=\pi k$, $k\in Z$.

2) $2\sin x-1=0$, $\sin x=\dfrac{1}{2}$, $x_2=(-1)^k\dfrac{\pi}{6}+\pi k$, $k\in Z$.

Endi $x\in[0;2\pi]$ bo'lgan yechimlarini topamiz:

$n=-1$ da $x_1=-\pi$, $x_2=-\dfrac{\pi}{6}-\pi$ yechimlar berilgan kesmaga tegishli emas.

$n=0$ da $x_1=0\in[0;2\pi]$, $x_2=\dfrac{\pi}{6}\in[0;2\pi]$,

$n=1$ da $x_1=\pi \in[0;2\pi]$, $x_2=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}\in[0;2\pi]$,

$n=2$ da $x_1=2\pi \in[0;2\pi]$, $x_2=\frac{\pi}{6}+2\pi=\frac{13\pi}{6}\notin[0;2\pi]$.

Shunday qilib , tenglamaning berilgan kesmadagi ildizlari yig‘indisi quyidagicha:

$$0+\frac{\pi}{6}+\pi+\frac{5\pi}{6}+2\pi=4\pi \ .$$

Javob: 4π.

10-misol. $8^{\sin^2 x}-2^{\cos^2 x}=0$ **tenglamani yeching.**

Yechish: Berilgan tenglamani ko‘rsatkichli tenglama deb qarab, bir xil asosga keltiramiz.

$$(2^3)^{\sin^2 x}=2^{\cos^2 x}$$

yoki

$$2^{3\cdot\sin^2 x}=2^{\cos^2 x}.$$

Ma’lumki, $\cos^2 x=1-\sin^2 x$. Ushbu ayniyatdan foydalanib, yuqoridagi tenglamani quyidagi ko‘rinishda yozib olamiz:

$$2^{3\cdot\sin^2 x}=2^{1-\sin^2 x}.$$

Endi darajalarini tenglashtiramiz:

$$3\sin^2 x=1-\sin^2 x,$$

Tenglamani $\sin^2 x$ ga nisbatan yechamiz.

$$4\sin^2 x=1.$$

Tenglamaning ikkala tomonini 4 ga bo‘lamiz.

$$\sin^2 x=\frac{1}{4}$$

yoki

$$\sin x=\pm\frac{1}{2}.$$

(2) ga asosan quyidagini yozamiz:

$$\left[\begin{array}{l} x=(-1)^k \arcsin\dfrac{1}{2}+\pi\cdot k \ ,k\in Z \\ x=(-1)^k \arcsin\left(-\dfrac{1}{2}\right)+\pi\cdot k \ ,k\in Z. \end{array}\right.$$

Ma'lumki, $\arcsin\frac{1}{2}=\frac{\pi}{6}$ va $\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$.

$$\left[\begin{array}{l} x=(-1)^k \dfrac{\pi}{6}+\pi\cdot k \ ,k\in Z \\ x=(-1)^{k+1} \dfrac{\pi}{6}+\pi\cdot k \ ,k\in Z. \end{array}\right.$$

Bu yechimlarni quyidagi bitta tenglik yordamida yozish ham mumkin:

$$x=\pm\frac{\pi}{6}+\pi k \ , \ k\in Z \ .$$

Javob: $x=\pm\frac{\pi}{6}+\pi k \ , \ k\in Z$ **.**

3. $tgx=a$ **tenglama.**

Har qanday a da $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ intervalda aniq bitta shunday x mavjudki, $tgx=a$ bo'ladi, bu $arctga$ dir. Shu sababli

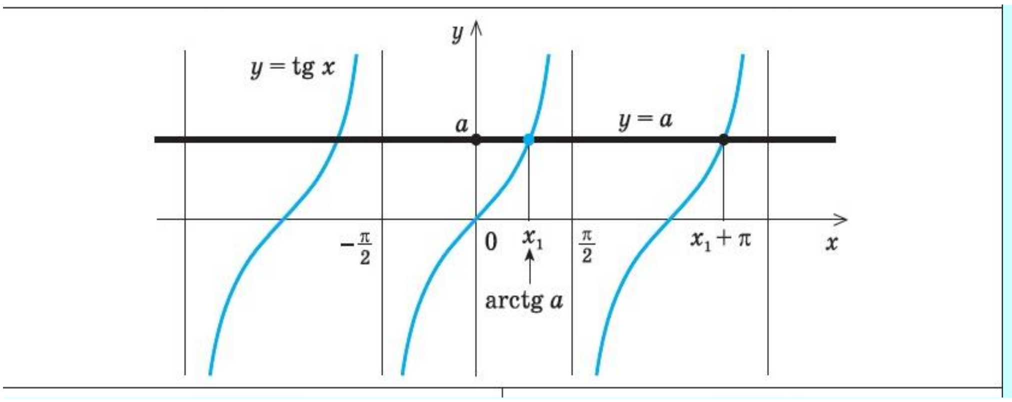
$$tgx=a$$

tenglama uzunligi π ga teng $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ intervalda aniq bitta ildizga ega.

Tangens - davri π ga teng davriy funksiya bo'lgani uchun tenglamaning qolgan ildizlari topilgan $arctga$ ildizdan $\pi n, n\in Z$ ga farq qiladi , ya'ni

$$x=arctga+\pi n \ , \ n\in Z \qquad \textbf{(3)}$$

$tgx=a$ tenglamaning yechimlarini quyidagicha tasvirlash mumkin:



11-misol. $tgx = \sqrt{3}$ **tenglamani yeching.**

Yechish: **(3)** ga ko‘ra

$$x = arctg\sqrt{3} + \pi n, n \in Z,$$

$arctg\sqrt{3} = \frac{\pi}{3}$ bo‘lgani uchun quyidagi

$$x = \frac{\pi}{3} + \pi n, n \in Z$$

yechimlarni keltirib chiqaramiz.

Javob: $x = \frac{\pi}{3} + \pi n, n \in Z$

12-misol. $tgx = 5{,}177$ **tenglamani taqribiy yeching .**

Yechish: **(3)** ga ko‘ra

$$x = arctg5{,}177 + \pi n, n \in Z$$

ekani kelib chiqadi.

Kalkulyator yordamida topamiz:

$$arctg5{,}177 \approx 1{,}3800.$$

Demak ,

$$x \approx 1{,}38 + \pi n, n \in Z.$$

taqribiy yechimlarni keltirib chiqaramiz.

Javob: $x \approx 1{,}38 + \pi n, n \in Z.$

13-misol. $ctgx = -\sqrt{3}$ **tenglamani yeching.**

Yechish: Ma’lumki , $ctgx = \frac{1}{tgx}$.

Shuning uchun berilgan tenglamani quyidagi ko‘rinishda yozamiz:

$$tgx = -\frac{1}{\sqrt{3}}.$$

(3) ga ko‘ra $x = arctg\left(-\frac{1}{\sqrt{3}}\right) + \pi n\ ,\ n \in Z,$

Tenglamadagi $arctg\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$ almashtiramiz.

$$x=-\frac{\pi}{6}+\pi n\ ,\ n\in Z\ .$$

yechimlarni keltirib chiqaramiz.

Javob: $x=-\frac{\pi}{6}+\pi n\ ,\ n\in Z.$