PROVING INEQUALITIES USING THE LAGRANGE FUNCTION

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Abstract. *Proving inequalities has a special place in the International Mathematical Olympiad. There are various methods of proving such inequalities, and this article presents a method of proving inequalities using the Lagrange function, which is simple and easy to use.*

Key words. *Lagrange function, conditional extremum, sufficiency conditions of extremum.*

It is known that the issue of finding extremums of multivariable functions is relevant in practice. Also, the extremum of a multivariable function found under certain conditions is important in solving optimization problems and in practice. In particular, the finding of conditional extremums of functions of this type can be used to prove inequalities in mathematical Olympiad Problems. This article presents an understanding of a conditional extremum and the Lagrange function, as well as a method of proving the inequalities of the International Mathematical Olympiad using this function.

Definition. Suppose that the function $f(x)$ with n variables is defined around a point a. If we find that the circumference of point a is such that the inequality $f\ (a)\ \geq f\ (x)\ (f\ (a)\ \leq f\ (x))$ holds for any point x taken from this circumference, then the function f is the *local maximum (local minimum)* at point $a \in \mathbb{R}^n$.

We now study the problem of finding the extreme values of a given function when some additional conditions are met. In this case, the additional conditions are usually given in the form of limiting the values of the variables.

For example, the arguments for the maximum value of the function $f(x, y, z)$ with three variables

$$g(x, y, z) = 0$$

let us consider the problem of finding when an additional condition is satisfied. In this case, the function of three variables $g(x, y, z)$ is a function that is sufficiently differentiable. In this case, the maximum value is called the *conditional maximum*, the minimum value is called the *conditional minimum*. Conditional maximum and conditional minimum together are called *conditional extremum*.

The Lagrange multiplier method is mainly used to find conditional extrema. That is, by selecting such a number μ, the following Lagrange function is created:

$$L(x, y, z, \mu) = f(x, y, z) - \mu g(x, y, z)$$

Then the following system of equations, formed by equalizing all partial derivatives of the Lagrange function to zero, is solved and the number μ is found:

$$\begin{cases} \dfrac{\partial L(x, y, z, \mu)}{\partial x} = 0 \\ \dfrac{\partial L(x, y, z, \mu)}{\partial y} = 0 \\ \dfrac{\partial L(x, y, z, \mu)}{\partial z} = 0 \end{cases}$$

Putting the found number μ in the system of equations, we find the relationship between the variables, conditioning the found relations $g(x, y, z) = 0$, which gives the function $f(x, y, z)$ to the conditional extremum (x, y, z) point is found. The value of the found point in the function $f(x, y, z)$ is the conditional extremum of the function. Sufficient conditions of the extremum are used to determine whether the found point is a conditional maximum or a conditional minimum.

Below are the inequalities that can be proved using this method.

Example 1 [Korea-1998]. If x, y, z are positive numbers and satisfy the equality

$$x + y + z = xyz,$$

then prove the following inequality:

$$\frac{1}{\sqrt{1 + x^2}} + \frac{1}{\sqrt{1 + y^2}} + \frac{1}{\sqrt{1 + z^2}} \leq \frac{3}{2}.$$

Proof.

First, let's look at the following function: $f(x, y, z) = \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}}$.

According to the condition of the problem, when the arguments of this function satisfy the condition $\varphi(x, y, z) = xyz - x - y - z = 0$, we must find the conditional maximum value of the function $f(x, y, z)$.

We construct the Lagrangian function:

$$L(x,y,z,\lambda) = f(x,y,z) - \lambda\varphi(x,y,z)$$
$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} - \lambda(xyz - x - y - z).$$

here $x, y, z \in \mathbb{R}^+$.

Using partial derivatives of the constructed Lagrange function, we create the following system of equations:

$$\begin{cases} \frac{\partial L(x,y,z,\lambda)}{\partial x} = -\frac{x}{\sqrt{(1+x^2)^3}} - \lambda(yz-1) = 0 \\ \frac{\partial L(x,y,z,\lambda)}{\partial y} = -\frac{y}{\sqrt{(1+y^2)^3}} - \lambda(xz-1) = 0 \\ \frac{\partial L(x,y,z,\lambda)}{\partial z} = -\frac{z}{\sqrt{(1+z^2)^3}} - \lambda(xy-1) = 0 \end{cases}$$

From this system of equations can be found: $x = y = z = \sqrt{3}\,, \lambda = -\frac{\sqrt{3}}{16}$.

We check the found point $A(\sqrt{3}, \sqrt{3}, \sqrt{3})$ for the second sufficiency condition of the extremum. For this, we find the second-order differential of the Lagrange function:

$$d^2L = \frac{1-2x^2}{\sqrt{(1+x^2)^5}}dx^2 + \frac{1-2y^2}{\sqrt{(1+y^2)^5}}dy^2 + \frac{1-2z^2}{\sqrt{(1+z^2)^5}}dz^2 + 2\lambda z dxdy$$
$$+ 2\lambda y dxdz + 2\lambda x dydz.$$

According to the condition, $\varphi(x,y,z) = xyz - x - y - z = 0$. We find the second-order differential of the function $\varphi(x,y,z)$:

$$d^2\varphi(x,y,z) = 2(xdydz + ydxdz + zdxdy) = 0,$$

Thus,

$$d^2L = \frac{1-2x^2}{\sqrt{(1+x^2)^5}}dx^2 + \frac{1-2y^2}{\sqrt{(1+y^2)^5}}dy^2 + \frac{1-2z^2}{\sqrt{(1+z^2)^5}}dz^2.$$

If we put $A(\sqrt{3}, \sqrt{3}, \sqrt{3})$ in the last equation, then we have

$$d^2L|_A = -\frac{5}{32}dx^2 - \frac{5}{32}dy^2 - \frac{5}{32}dz^2 < 0,$$

Thus, the function $f(x,y,z)$ reaches a conditional maximum at the point $A(\sqrt{3}, \sqrt{3}, \sqrt{3})$, and from that we have

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{2}.$$

Example 2 [United Kingdom-1999]. If the equality $x+y+z=1$ holds for non-negative numbers x,y,z, prove the following inequality:

$$7(xy+yz+xz)\leq 2+9xyz.$$

Proof.

First, let's look at the following function: $f(x,y,z)=7(xy+yz+xz)-9xyz$.

According to the condition of the problem, when the arguments of this function satisfy the condition $\varphi(x,y,z)=x+y+z-1=0$, we must find the conditional maximum value of the function $f(x,y,z)$.

We construct the Lagrange function:

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda\varphi(x,y,z)$$
$$=7(xy+yz+xz)-9xyz-\lambda(x+y+z-1).$$

Using partial derivatives of the constructed Lagrange function, we create the following system of equations:

$$\begin{cases}\dfrac{\partial L(x,y,z,\lambda)}{\partial x}=7(y+z)-9yz-\lambda=0\\ \dfrac{\partial L(x,y,z,\lambda)}{\partial y}=7(x+z)-9xz-\lambda=0\\ \dfrac{\partial L(x,y,z,\lambda)}{\partial z}=7(x+y)-9xy-\lambda=0\end{cases}$$

From this system of equations can be found: $x=y=z=\frac{1}{3}, \lambda=\frac{11}{3}$.

We check the found point $A\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ for the second sufficiency condition of the extremum. For this, we find the representation of the second-order differential of the function $L(x,y,z,\lambda)$ at the found point $A\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$:

$$d^2L|_A=[2(7-9z)dxdy+2(7-9y)dxdz+2(7-9x)dydz]|_A$$
$$=8(dxdy+dxdz+dydz).$$

According to the condition of the problem, we have $\varphi(x,y,z)=x+y+z-1=0$. We calculate the second-order differential of the square of this function:

$$\varphi^2(x,y,z)=x^2+y^2+z^2+2xy+2xz+2yz-2x-2y-2z+1=0,$$
$$d^2\varphi^2(x,y,z)=2dx^2+2dy^2+2dz^2+4dxdy+4dxdz+4dydz=0.$$

From this we find the following equality:

$$dxdy+dxdz+dydz=-\frac{1}{2}(dx^2+dy^2+dz^2).$$

Thus,

$$d^2L=8(dxdy+dydz+dxdz)=-4(dx^2+dy^2+dz^2)<0.$$

The function $f(x,y,z)$ reaches a conditional maximum at the point $A(\sqrt{3},\sqrt{3},\sqrt{3})$, and from that we have

$$7(xy+yz+xz) \leq 2+9xyz.$$

Example 3 [Hungary – 1996]. If the sum of positive numbers x,y is equal to 1, then prove the following inequality:

$$\frac{x^2}{x+1}+\frac{y^2}{y+1} \geq \frac{1}{3}.$$

Proof.

First, let's look at the following function: $f(x,y)=\frac{x^2}{x+1}+\frac{y^2}{y+1}$.

According to the condition of the problem, when the arguments of this function satisfy the condition $\varphi(x,y)=x+y+1=0$, we must find the conditional minimum value of the function $f(x,y)$.

We construct the Lagrange function:

$$L(x,y,\lambda)=f(x,y)-\lambda\varphi(x,y)=\frac{x^2}{x+1}+\frac{y^2}{y+1}-\lambda(x+y-1).$$

here $x,y \in \mathbb{R}^+$.

Using partial derivatives of the constructed Lagrange function, we create the following system of equations:

$$\begin{cases} \dfrac{\partial L(x,y,\lambda)}{\partial x}=1-\dfrac{1}{(1+x)^2}-\lambda=0 \\ \dfrac{\partial L(x,y,\lambda)}{\partial y}=1-\dfrac{1}{(1+y)^2}-\lambda=0 \end{cases}$$

From this system of equations can be found: $x=y=\frac{1}{2}, \lambda=\frac{5}{9}$.

We check the found point $A\left(\frac{1}{2},\frac{1}{2}\right)$ for the second sufficiency condition of the extremum. For this, we find the representation of the second-order differential of the function $L(x,y,\lambda)$ at the found point $A\left(\frac{1}{2},\frac{1}{2}\right)$:

$$d^2L=\frac{2}{(1+x)^3}dx^2+\frac{2}{(1+y)^3}dy^2 \Rightarrow d^2L|_A=\frac{16}{27}dx^2+\frac{16}{27}dy^2>0.$$

Thus, the function $f(x,y)$ reaches a conditional minimum at the point $A\left(\frac{1}{2},\frac{1}{2}\right)$, and from that we have

$$\frac{x^2}{x+1}+\frac{y^2}{y+1} \geq \frac{1}{3}.$$

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