A METHOD OF FINDING THE SOLUTION OF SOME IMPORTANT DIFFERENTIAL EQUATIONS

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Abstract. *It is known that the mathematical model of many processes in our life is represented by differential equations, and it is important to find the solutions of these differential equations in a simpler way. This article presents a method of solving some important differential equations by expanding the solution into a power series.*

Keywords. *power series, recurrent sequence, Gauss equation, Cylindrical function, Gegenbauer equation, Elliptic integrals.*

Below is a method of finding the general solution of differential equations by expanding this solution into a power series. Equations of this type are used in many areas of mathematical physics, including the study of some issues of quantum mechanics. Although the equations are simple differential equations with variable coefficients, they cannot be solved using elementary functions.

1. Gauss equation.

Let us be given the following differential equation

$$x(x-1)y'' + \big((\alpha+\beta+1)x-\gamma\big)y' + \alpha\beta y = 0 \qquad (1)$$

This equation (1.1) is called hypergeometric equation or Gaussian equation. Gauss equation is defined symmetrically with respect to α and β. Let's consider the problem of solving this equation.

Below we deal with finding the solution of equation (1) that satisfies the initial condition $y(0) = 1$. Let's assume that the solution of equation (1) is expanded in the following form (in equation (1) and in the other problems given below, the assumption made is valid because the function $y(x)$ and its derivatives are continuous in $\mathbb{R}$):

$$y(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \qquad (2)$$

$\{a_n\}$ is an assigned sequence and the equality $a_0 = y(0) = 1$ holds. We study the problem of finding the general term of the sequence $\{a_n\}$. It is known that power series can be differentiated in the field of convergence (this assertion is also used in the other problems). So the following equations are valid

$$y'(x) = \sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!}, y''(x) = \sum_{n=0}^{\infty} \frac{a_{n+2}x^n}{n!} \tag{3}$$

We simplify equations (2) and (3) by putting them into equation (1):

$$x(x-1)\sum_{n=0}^{\infty}\frac{a_{n+2}x^n}{n!} + ((\alpha+\beta+1)x-\gamma)\sum_{n=0}^{\infty}\frac{a_{n+1}x^n}{n!} + \alpha\beta\sum_{n=0}^{\infty}\frac{a_nx^n}{n!} = 0$$

$$\sum_{n=0}^{\infty}\frac{a_{n+2}x^{n+2}}{n!} - \sum_{n=0}^{\infty}\frac{a_{n+2}x^{n+1}}{n!} + (\alpha+\beta+1)\sum_{n=0}^{\infty}\frac{a_{n+1}x^{n+1}}{n!} - \gamma\sum_{n=0}^{\infty}\frac{a_{n+1}x^n}{n!} +$$

$$+\alpha\beta\sum_{n=0}^{\infty}\frac{a_nx^n}{n!} = 0$$

It is necessary and sufficient that the sum of the coefficients in front of the corresponding powers of the unknowns is equal to zero for the resulting equality to be valid.Thus,

$$\frac{a_n}{(n-2)!} - \frac{a_{n+1}}{(n-1)!} + \frac{a_n}{(n-1)!}(\alpha+\beta+1) - \frac{a_{n+1}}{n!}\gamma + \frac{a_n}{n!}\alpha\beta = 0$$

In this case, by condensing similar terms, we get the following recurrent formula:

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+\gamma)}a_n.$$

The general term of the sequence $\{a_n\}$ can be found using the resulting recurrent formula and $a_0 = 1$.

$$a_n = \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)}{\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)}.$$

We put this found expression into equation (2):

$$y(x) = \sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)\beta(\beta+1)\cdot\ldots\cdot(\beta+n-1)\cdot x^n}{n!\cdot\gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1)}.$$

This resulting expression is a holomorphic solution of Gauss equation.

2. Cylindrical function.

Let us be given the following second-order differential equation

$$xy'' + 2\gamma y' + xy = 0 \tag{4}$$

If function $y = y(x)$ satisfying this differential equation, then it is called a cylindrical function.

Suppose that the solution of the differential equation (4) has the following form:

$$y(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}, \quad (5)$$

here $y(0) = a_0$, $y'(0) = 0$. In that case,

$$y'(x) = \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!}, \quad y''(x) = \sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!},$$

Based on equality (4), the following equalities can be written:

$$x \sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!} + 2\gamma \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} + x \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{a_{n+2} x^{n+1}}{n!} + 2\gamma \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n!} = 0,$$

$$\sum_{n=1}^{\infty} \frac{a_{n+1}}{(n-1)!} x^n + 2\gamma \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} + \sum_{n=1}^{\infty} \frac{a_{n-1} x^n}{(n-1)!} = 0$$

$$\Rightarrow 2\gamma a_1 + \sum_{n=1}^{\infty} \left[\frac{a_{n+1}}{(n-1)!} + 2\gamma \frac{a_{n+1}}{(n)!} + \frac{a_{n-1}}{(n-1)!} \right] x^n = 0.$$

The following equality is necessary and sufficient for the last condition to be valid for any x:

$$\frac{a_{n+1}}{(n-1)!} + 2\gamma \frac{a_{n+1}}{(n)!} + \frac{a_{n-1}}{(n-1)!} = 0.$$

Simplifying this equation, we get the following recurrence relation:

$$a_n = -\frac{(n-1)a_{n-2}}{(n+2\gamma-1)}.$$

Based on this recurrence formula, we can find the general term of the sequence $\{a_n\}$:

$$a_n = \begin{cases} (-1)^n \dfrac{(2n-1)(2n-3) \cdot \ldots \cdot a_0}{(2n+2\gamma-1)(2n+2\gamma-3) \ldots (2\gamma+1)}, & n = 2k \\ 0, & n = 2k-1 \end{cases}.$$

It means that all the odd-numbered terms of the searched sequence are equal to zero, and the even-numbered terms are different from zero. If we put the general term of the found sequence $\{a_n\}$ in (5), then we have

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)(2n-3) \cdot \ldots \cdot a_0 \cdot x^{2n}}{(2n)! \, (2n+2\gamma-1)(2n+2\gamma-3) \ldots (2\gamma+1)},$$

or

$$y(x)=\sum_{n=0}^{\infty}\frac{(-1)^n a_0}{n!\left(\gamma+\frac{1}{2}\right)\left(\gamma+\frac{3}{2}\right)\dots\left(\gamma+\frac{2n-1}{2}\right)}\cdot\left(\frac{x}{2}\right)^{2n}.$$

This resulting expression is a solution of the cylindrical function satisfying the conditions $y(0)=a_0, y'(0)=0$.

3. Gegenbauer equation.

Let us be given the following second-order differential equation

$$(x^2-1)y''-\big((2\alpha+1)x\big)y'+\alpha(\alpha+2\beta)y=0 \qquad (6)$$

This equation is called the Gegenbauer equation. Below we deal with finding the solution of equation (6) that satisfies the initial conditions $y(0)=1, y'(0)=0$.

Let us assume that the solution of equation (6) satisfying the initial conditions is as follows:

$$y(x)=\sum_{n=0}^{\infty}\frac{a_n x^n}{n!} \qquad (7)$$

Here, $\{a_n\}-$ is a sequence, and the equalities $y(0)=a_0=1, y'(0)=a_1=0$ are valid. We formally calculate the first and second order derivatives of the function (7) as follows:

$$y'(x)=\sum_{n=0}^{\infty}\frac{a_{n+1}x^n}{n!},\qquad y''(x)=\sum_{n=0}^{\infty}\frac{a_{n+2}x^n}{n!} \qquad (8)$$

If we put these (8) equations to equation (6), we have

$$(x^2-1)\sum_{n=0}^{\infty}\frac{a_{n+2}x^n}{n!}-(2\alpha+1)x\sum_{n=0}^{\infty}\frac{a_{n+1}x^n}{n!}+\alpha(\alpha+2\beta)\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}=0,$$

$$\sum_{n=0}^{\infty}\frac{a_{n+2}x^{n+2}}{n!}-\sum_{n=0}^{\infty}\frac{a_{n+2}x^n}{n!}-(2\alpha+1)\sum_{n=0}^{\infty}\frac{a_{n+1}x^{n+1}}{n!}+\alpha(\alpha+2\beta)\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}=0.$$

As in the above example, the sum of the coefficients in the corresponding powers of the variable equal to zero. Then we will have following recurrent formula:

$$\frac{a_n}{(n-2)!}-\frac{a_{n+2}}{n!}-(2\alpha+1)\frac{a_n}{(n-1)!}+\alpha(\alpha+2\beta)\frac{a_n}{n!}=0,$$

$$a_n=a_{n-2}\big((n-\alpha-2)^2+2(\alpha\beta+2-n)\big).$$

Using the found recurrent formula, the general term of the sequence $\{a_n\}$ can be found:

$$a_n=\begin{cases}(\alpha^2+2\alpha\beta)\big((\alpha-2)^2+2(\alpha\beta-2)\big)\dots\big((2n-\alpha-2)^2+2(\alpha\beta+2-2n)\big), & n=2k\\ 0 & ,n=2k-1\end{cases}$$

Using these results, the general representation of the desired function can be written as:

$$y(x) = \sum_{k=0}^{\infty} \frac{(\alpha^2 + 2\alpha\beta)\big((\alpha-2)^2 + 2(\alpha\beta - 2)\big)\dots\big((2n-\alpha-2)^2 + 2(\alpha\beta + 2 - 2n)\big)}{(2n)!} x^{2k}.$$

4. Elliptic integrals.

Integrals given in the following form are called first and second full elliptic integrals respectively:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi, \qquad F(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\varphi}} d\varphi \tag{9}$$

It is known that these integrals are existing and finite integrals, each of which is a function of the parameter k $(0 \le k < 1)$. We consider the problem of finding the expression of these functions in the form of a power series, and in this we use the fact that the functions we are looking for satisfy the following equations:

$$E''(k) + \frac{1}{k}E' + \frac{E(k)}{1-k^2} = 0, \quad E'(k) = \frac{E(k) - F(k)}{k} \tag{10}$$

Let us assume that the function $E(k)$ is expanding into a power series in the following form:

$$E(k) = \sum_{n=0}^{\infty} \frac{a_n k^n}{n!} \tag{11}$$

Then, these equalities are valid:

$$E'(k) = \sum_{n=0}^{\infty} \frac{a_{n+1} k^n}{n!}, E''(k) = \sum_{n=0}^{\infty} \frac{a_{n+2} k^n}{n!} \tag{12}$$

We put equalities (11) and (12) to (10), then we have

$$\sum_{n=0}^{\infty} \frac{a_{n+2} k^n}{n!} + \frac{1}{k}\sum_{n=0}^{\infty} \frac{a_{n+1} k^n}{n!} + \frac{1}{1-k^2}\sum_{n=0}^{\infty} \frac{a_n k^n}{n!} = 0,$$

$$k(1-k^2)\sum_{n=0}^{\infty} \frac{a_{n+2} k^n}{n!} + (1-k^2)\sum_{n=0}^{\infty} \frac{a_{n+1} k^n}{n!} + k\sum_{n=0}^{\infty} \frac{a_n k^n}{n!} = 0,$$

$$\sum_{n=0}^{\infty} \frac{a_{n+2} k^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{a_{n+2} k^{n+3}}{n!} + \sum_{n=0}^{\infty} \frac{a_{n+1} k^n}{n!} - \sum_{n=0}^{\infty} \frac{a_{n+1} k^{n+2}}{n!} + \sum_{n=0}^{\infty} \frac{a_n k^{n+1}}{n!} = 0,$$

$$\sum_{n=-1}^{\infty}\frac{a_{n+1}k^n}{(n-1)!}+\sum_{n=0}^{\infty}\frac{a_{n+1}k^n}{n!}=\sum_{n=-3}^{\infty}\frac{a_{n-1}k^n}{(n-3)!}+\sum_{n=-2}^{\infty}\frac{a_{n-1}k^n}{(n-1)!}-\sum_{n=-1}^{\infty}\frac{a_{n-1}k^n}{(n-1)!}.$$

For the power series to be equal, it is necessary and sufficient that the coefficients in front of the corresponding powers of the variable are equal. Thus,

$$\frac{a_{n+1}}{(n-1)!}+\frac{a_{n+1}}{n!}=\frac{a_{n-1}}{(n-3)!}+\frac{a_{n-1}}{(n-2)!}-\frac{a_{n-1}}{(n-1)!}.$$

From above formed a recurring formula of this form:

$$a_{n+1}=\frac{n^2(n-2)}{(n+1)}a_{n-1}.$$

Thus,

$$a_n=\frac{(n-1)^2(n-3)}{n}a_{n-2}.$$

We simplify the recurrence formula using the relation: $a_0=E(0)=\frac{\pi}{2},\ a_1=E'(0)=0.$

In that case

$$a_2=\frac{(2-1)^2}{2}(-1)\pi\,,a_3=0,\ \ a_4=-\frac{9}{16}\pi,\dots$$

$$a_{(2n-1)}=0,\qquad a_{2n}=-\frac{\pi}{2}\frac{\left((2n-1)!!\right)^2(2n-3)!!}{(2n)!!}.$$

Thus, the following equality holds:

$$E(k)=\frac{\pi}{2}\left(1-\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{k^{2n}}{2n-1}\right).$$

Using this equality and the second equality presented in (10), we have the following relations:

$$\sum_{n=0}^{\infty}\frac{a_{n+1}k^n}{n!}=\frac{1}{k}\sum_{n=0}^{\infty}\left(\frac{a_nk^n}{n!}-\frac{b_nk^n}{n!}\right),\quad \frac{a_n}{(n-1)!}=\frac{a_n-b_n}{n!},\quad b_n=-a_n(n-1).$$

From these, the following equality is formed:

$$b_{2n}=-a_{2n}(2n-1),\qquad b_{2n}=\frac{\pi}{2}\frac{\left((2n-1)!!\right)^3}{(2n)!!}.$$

Thus, the following equality is valid for the second full elliptic integral:

$$F(k)=\sum_{n=0}^{\infty}\frac{b_nk^n}{n!}=\frac{\pi}{2}\left(1+\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2k^{2n}\right).$$

Explanation. It should be said that it is desirable to (approximate) calculate full elliptic integrals only for sufficiently small values of k using the above series**.**

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