GYOLDER TENGIZLIGI VA UNING MASALALARDA QO‘LLANILISHI

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ANNOTATSIYA

Ushbu maqolada muhim tengsizlardan biri hisoblangan Gyolder tengsizligi va uning qo‘llanilishiga doir masalalar va ularning yechimlari batafsil tushuntirilgan. Mustaqil yechish uchun yetarlicha masalalar berilgan.

Kalit so‘zlar: *AM-GM tengsizligi, Koshi-Shvartz tengsizligi, Gyolder tengsizligi.*

ABSTRACT

In this article, Holder’s inequality, which is considered one of the most important inequalities, and its application problems and their solutions are explained in detail. Sufficient problems are given for independent solution.

Key words: *AM-GM inequality, Cauchy-Schwarz inequality, Golder inequality.*

АННОТАЦИЯ

В этой статье подробно объясняется неравенство Гёльдера, которое считается одним из важнейших неравенств, а также проблемы, связанные с его применением и их решением. Приведено достаточное количество задач для самостоятельного решения.

Ключевые слова: *Неравенство AM-GM, неравенство Коши-Шварца, неравенство Голдера.*

Ushbu maqolani o‘rganishdan avval, o‘quvchining quyidagi tengsizliklar bilan tanish bo‘lishi tavsiya etiladi. Bular sodda tengsizliklar [1], AM-GM tengsizligi [1] va Koshi-Shvartz tengsizliklari [2]. Ularni ko‘rsatilgan manbalardan ko‘rib chiqishingiz mumkin.

Dastlab Koshi-Shvartz tengsizligini keltiramiz.

Teorema 1. (Koshi-Shvartz tengsizligi) $a_1,a_2,...,a_n,b_1,b_2,...,b_n,c_1,c_2,...,c_n$ haqiqiy sonlari berilgan bo‘lsin. U holda ushbu tengsizlik o‘rinli:

$$(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2)\geq(a_1b_1+a_2b_2+...+a_nb_n)^2$$

Bu tengsizlikda tenglik holi faqat va faqat $\frac{a_1}{b_1}=\frac{a_2}{b_2}=...=\frac{a_n}{b_n}$ bo‘lganda bajariladi.

Ko‘rishimiz mumkinki Koshi-Shvartz tengsizligining chap tarafida ikkita ko‘paytuvchi va o‘zgaruvchilar ikkinchi darajaga oshirilgan. Gyolder tengsizligida ikki ko‘paytuvchini umumlashtiramiz. Masalan haqiqiy $a_1,a_2,...,a_n,b_1,b_2,...,b_n,c_1,c_2,...,c_n$ sonlari uchun, gyolder tengsizligiga ko‘ra quyidagi tengsizlik o‘rinli:

$$(a_1^3+a_2^3+...+a_n^3)(b_1^3+b_2^3+...+b_n^3)(c_1^3+c_2^3+...+c_n^3)\geq(a_1b_1c_1+a_2b_2c_2+...+a_nb_nc_n)^3$$

Endi Gyolder tengsizligini umumiy ko‘rinishini keltiramiz:

Teorema 2. (Gyolder tengsizligi [3]) Barcha $a_{i_j}>0$ sonlar uchun quyidagi tengsizlik o‘rinli:

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n}a_{i_j}\right)\geq\left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m}a_{i_j}}\right)^m$$

Gyolder tengsizligining $m=2$ bo‘lgan holi Koshi-Shvartz tengsizligini ifodalaydi.

Endi ushbu tengsizlikni misollarda qo‘llanishlarini ko‘rib o‘tamiz.

Masala 1. a,b,c musbat haqiqiy sonlar berilgan. U holda quyidagi tengsizlikni isbotlang:

$$(a^3+2)(b^3+2)(c^3+2)\geq(a+b+c)^3$$

Yechim. Gyolder tengsizligi ko‘ra

$$(a^3+1+1)(1+b^3+1)(1+1+c^3)\ge\left(\sqrt[3]{a^3\cdot1\cdot1}+\sqrt[3]{1\cdot b^3\cdot1}+\sqrt[3]{1\cdot1\cdot c^3}\right)^3$$

Yoki

$$(a^3+2)(b^3+2)(c^3+2)\ge(a+b+c)^3. \blacksquare$$

Masala 2. (Junior Balkan MO 2002) Barcha a,b,c musbat haqiqiy sonlari uchun quyidagi tengsizlikni isbotlang:

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\ge\frac{27}{2(a+b+c)^2}$$

Yechim. Bu masala Gyolder tengsizligi qo‘llanilib yechiladigan masalalarga yorqin misol bo‘ladi. Dastlab $3^3=27$ va $2(a+b+c)=(a+b)+(b+c)+(c+a)$ tengliklarni hisobga olamiz.

Tengsizlikni ikkala tomoniga ham $2(a+b+c)^2$ ifodani ko‘paytirib, yuqoridagi tenglikka ekvivalent bo‘lgan ushbu tengsizlikni hosil qilamiz:

$$((a+b)+(b+c)+(c+a))(b+c+a)\left(\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\right)\ge27$$

Bu tengsizlik esa Gyolder (m=3) tengsizligiga ko‘ra o‘rinli. Bundan esa

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\ge\frac{27}{2(a+b+c)^2}$$

Tengsizlik ham o‘rinlli ekanligi kelib chiqadi. ■

Masala 3. $a+b=1$ tenglikni qanoatlantiradigan a,b musbat haqiqiy sonlari uchun quyidagi tengsizlikni isbotlang:

$$\frac{1}{a^2}+\frac{1}{b^2}\ge8$$

Yechim. Shartga ko‘ra, ko‘rishimiz mumkinki,

$$\frac{1}{a^2}+\frac{1}{b^2}=(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

So‘ngra Gyolder (m=3) tengsizligini qo‘llasak,

$$(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\left(\sqrt[3]{\frac{a\cdot a}{a^2}}+\sqrt[3]{\frac{b\cdot b}{b^2}}\right)^3=8\ .\ ■$$

Masala 4. $a+b+c=1$ tenglikni qanoatlantiradigan a,b,c musbat haqiqiy sonlari uchun quyidagi tengsizlikni isbotlang:

$$4a^3+9b^3+36c^3\geq 1$$

Yechim. Bilamizki,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1.$$

Endi Gyolder (m=3) tengsizligini qo‘llasak,

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(4a^3+9b^3+36c^3\right)\geq\left(a+b+c\right)^3=1$$

Shu bilan isbot tugadi. ■

Masala 5. a,b,c musbat haqiqiy sonlar berilgan. Ular uchun quyidagi tengsizlikni isbotlang:

$$\frac{a+b}{\sqrt{a+2c}}+\frac{b+c}{\sqrt{b+2a}}+\frac{c+a}{\sqrt{c+2b}}\geq 2\sqrt{a+b+c}$$

Yechim. Dastlab Gyolder tengsizligini quyidagicha qo‘llaymiz:

$$\left(\frac{a+b}{\sqrt{a+2c}}+\frac{b+c}{\sqrt{b+2a}}+\frac{c+a}{\sqrt{c+2b}}\right)^2\left((a+b)(a+2c)+(b+c)(b+2a)+(c+a)(c+2b)\right)\geq 8(a+b+c)^3$$

Va yana bilamizki:

$$\left((a+b)(a+2c)+(b+c)(b+2a)+(c+a)(c+2b)\right)=(a+b+c)^2+3(ab+bc+ca)$$

Bundan kelib chiqadiki yuqoridagi tengsizlikni isbotash uchun quyidagi tengsizlikni isbotlash yetarli:

$$\frac{8(a+b+c)^3}{(a+b+c)^2+3(ab+bc+ca)}\geq\left(2\sqrt{a+b+c}\right)^2$$

Bu tengsizlik esa sodda bo‘lgan $(a+b+c)^2\geq 3(ab+bc+ca)$ yoki $a^2+b^2+c^2\geq ab+bc+ca$ tengsizliklari bilan ekvivalent. ■

Mustaqil yechish uchun masalalar:

Masala 6. a,b,c musbat haqiqiy sonlar berilgan. U holda quyidagilarni isbotlang:

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)^3}{3(ab+bc+ca)}$$

(b) $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{27}{ab+bc+ca}}$$

(c) $$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}$$

(d) $$\frac{a^2+b^2+c^2}{a+b+c}\geq\sqrt{\frac{abc(a+b+c)}{ab+bc+ca}}$$

Masala 7. a,b,c musbat haqiqiy sonlar berilgan bo‘lib ularning yig‘indisi 1 ga teng. Quyidagi tengsizlikni isbotlang:

$$\sqrt[3]{99}\geq\sqrt[3]{1+8a}+\sqrt[3]{1+8b}+\sqrt[3]{1+8c}$$

Masala 8. $a_1,a_2,...,a_n$ musbat haqiqiy sonlar uchun, ushbu tengsizlikni isbotlang:

$$(1+a_1)(1+a_2)\cdot...\cdot(1+a_n)\geq\left(1+\sqrt[n]{a_1a_2...a_n}\right)^n$$

Masala 9. a,b,c,x,y,z musbat haqiqiy sonlar uchun, ushbu tengsizlikni isbotlang:

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3(x+y+z)}$$

Masala 9. Yig‘indisi 1 gateng bo‘lgan a,b,c musbat haqiqiy sonlar uchun, ushbu tengsizlikni isbotlang:

$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)}\geq 4.5.$$

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