

CIRCLE MAPPINGS WITH BREAKS

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ABSTRACT

In this article, we study a one-parameter family of circle homeomorphisms with one break point. It is proved that in the case of a rational rotation number the number of periodic trajectories does not exceed two.

Key words: circle homeomorphism, renormalization, rotation number.

Consider a one-parameter family of mappings of the unit circle [1]:

$$T_{\Omega}x = \{f(x, \Omega)\}, \quad x \in S^1 = [0, 1), \quad \Omega \in [0; 1]$$

where the bracket $\{\cdot\}$ - denotes the fractional part of the number, and $f(x, \Omega)$ - satisfies the following conditions:

a) at a fixed Ω , $f(x; \Omega)$ - continuous monotonically increasing function;

b) $f(0; 0) = 0$, $f(x + 1; \Omega) = f(x; \Omega) + 1$, $x \in R^1$;

c) $\frac{\partial f(x; \Omega)}{\partial \Omega} > \text{const} > 0$;

d) $t_0 : [0; 1] \rightarrow [0; 1]$ continuous curve;

e) for every fixed $\Omega \in [0; 1]$, $\frac{\partial f(x; \Omega)}{\partial x} > \text{const} > 0$; for

$\forall x \in S^1 \setminus \{t_0(\Omega)\}$, $f(x; \Omega) \in C^{2+\varepsilon}(S^1 \setminus \{t_0(\Omega)\})$, at some $\varepsilon > 0$ and

$$\frac{f'_-(t_0(\Omega), \Omega)}{f'_+(t_0(\Omega), \Omega)} = c(\Omega) \neq 1.$$

Let us ρ_{Ω} denote the number of rotations corresponding to, responsible $T_{f_{\Omega}}$ [2]:

$$\rho_{\Omega} = \lim_{n \rightarrow \infty} \frac{f^{(n)}(x, \Omega)}{n}$$

From the $d) - e)$ conditions it follows that $T_{\Omega}x$ for each fixed value of the parameter has only one break point $t_0(\Omega)$. The number $c(\Omega)$ is called the break point T_{Ω} .

Everywhere below we will denote by the $f^{(n)}$ – n st superposition of the function f . It is easy to see that ρ_{Ω} monotonically (not strictly) depends on the parameter Ω .

Note that each rational $\rho = \frac{p}{q}$ corresponds to a non-degenerate segment (values

Ω such that $\rho_{\Omega} = \frac{p}{q}$, while irrational ρ corresponds only to Ω).

Let $A = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2} \right) \subset (0, 1)$ - be the Faria interval of the n – nd level [1]:

1) $p_2q_1 - p_1q_2 = 1$

2) All rational numbers inside the interval A have the form $\frac{kp_1 + lp_2}{kq_1 + lq_2}$.

Rational number with minimum denominator is $\frac{p_1 + p_2}{q_1 + q_2}$.

We choose an arbitrary point x_0 on the circle and a segment of the trajectory of this point $\{x_i = T_{\Omega}^i x_0, 0 \leq i < q_1 + q_2\}$. Denote $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ segments $[x_0, x_{q_1}]$ and $[x_{q_2}, x_0]$, respectively. We also denote the images of these segments under the action of T_{Ω} by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ [1]:

$$\Delta_i^{(1)} = T^i \Delta_0^{(1)}, \quad \Delta_j^{(2)} = T^j \Delta_0^{(2)} \quad .$$

The following assertion was proved in [2] and works without any changes in our situation.

Лемма 1. Suppose $\rho(T) \in \left(\frac{p_1}{q_1}, \frac{p_2}{q_2} \right)$. Trajectory segment

$\{x_i = T_{\Omega}^i x_0, 0 \leq i < q_1 + q_2\}$ divides the circle into non-intersecting segments $\Delta_i^{(1)}, 0 \leq i < q_2$ and $\Delta_j^{(2)}, 0 \leq j < q_1$.

Denote the constructed partition $\xi(A, x_0)$. Let's put $v = \text{var}_{S^1} \ln f' < \infty, \bar{v} = v + |\ln f'(x_0 - 0) + \ln f'(x_0 + 0)|$, $q = \max(q_1, q_2)$, $p = \max(p_1, p_2)$. Consider an arbitrary trajectory $y_i = T_{\Omega}^i y_0, y_0 \in S^1$ such that $y_i \neq x_0 = 0, 0 \leq i < q_2$.

Лемма 2. Suppose $\rho(T) \in \left(\frac{p_1 + p_2}{q_1 + q_2}, \frac{p}{q}\right)$ or $\rho(T) = \left(\frac{p}{q}\right)$. Then

$$e^{-\bar{v}} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{\bar{v}}.$$

Let $A_n = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$ be a Farey interval of rank $n - [1]$, and $A_m, m < n$ - be some

Farey interval of rank m containing A_n . Let $\rho(T) \in A_n$. Let's choose Δ_n - arbitrary element of partition $\xi(A_m, t_0)$ containing A_n . Let's denote by $|\Delta|$.

Лемма 3. Let's put $\lambda = (1 + e^{-\bar{v}})^{-\frac{1}{2}} < 1$.

$$|\Delta_n| \leq \text{const} \lambda^{n-m} |\Delta_m|, \quad |\Delta_n| \leq \text{const} \lambda^n.$$

Let the continued fraction expansion of ρ be of the form

$$\rho(f(x, \Omega)) = \frac{p}{q} = [k_1, k_2, \dots, k_n], k_n \geq 2.$$

Let's designate $I\left(\frac{p}{q}\right)$ the segment of the value of the parameter Ω such that

$$\rho(\Omega) = \frac{p}{q}. \text{ Fix some } \Omega \in I\left(\frac{p}{q}\right) \text{ and denote } f = f_{\Omega}, T_f = T_{f_{\Omega}}. \text{ For a rational rotation}$$

number of $\rho(\Omega) = \frac{p}{q}$, there always exists at least one periodic trajectory of period q .

Let $\{y^{(i)}, 0 \leq i \leq q - 1\}$ be an arbitrary periodic trajectory. Let $[y_1, y_2]$ denote the segment formed by the trajectory $\{y^{(i)}, 0 \leq i \leq q - 1\}$ and containing the singular point t_0 . Let's move on to renormalized coordinates:

$$x = y_2 + (y_1 - y_2)z$$

and define the function corresponding to T_f^q in the renormalized coordinate system:

$$\bar{f}(z) = \frac{1}{y_1 - y_2} [T_f^q(y_2 + (y_1 - y_2)z) - y_2], \quad z \in [0, 1].$$

Denote by d the renormalized coordinate of the point t_0 :

$$d = (t_0 - y_2)/(y_1 - y_2)$$

and define the function $F_d(z), z \in [0, 1]$:

$$F_d(z) = \begin{cases} \frac{zc}{d(1 - c^2) + c^2 + z(c - 1)}, & z \in [0, d] \\ \frac{d(1 - c^2) + zc^2}{d(1 - c^2) + c + zc(c - 1)}, & z \in [d, 1] \end{cases}$$

Theorem 1. There is a constant $c_3 > 0$ such that

$$\|\bar{f}(z) - F_d(z)\|_{C^2([0,1] \setminus \{d\})} \leq c_3 \lambda^{n\epsilon}. \quad (1)$$

Proof. Consider the partition of the circle generated by the trajectory $(y^{(i)}, 0 \leq i \leq q - 1)$. Denote $\Delta_0 = [y_1, y_2], \Delta_i = T_\theta^i \Delta_0, 1 \leq i \leq q - 1$. Obviously $T_\theta^q \Delta_0 = \Delta_0$. It is not difficult to show [1] that $|\Delta_i| \leq const \lambda^n, 1 \leq i \leq q - 1$. Function $\bar{f}(z)$ can be represented as a superposition of two functions f_1 and f_2 , corresponding to mappings $T_\theta : \Delta_0 \rightarrow \Delta_1, T_\theta^{q-1} : \Delta_1 \rightarrow \Delta_q = \Delta_0$. Let us determine the relative coordinates inside the segments Δ_i :

$$x = T_f^i y_2 + (T_f^i y_1 - T_f^i y_2)z.$$

Then the functions f_1 and f_2 can be written as:

$$f_1(z_0) = \frac{1}{(T_\theta y_1 - T_\theta y_2)} [T_\theta(y_2 + (y_1 - y_2)z_0) - T_\theta y_2]$$

$$f_2(z_1) = \frac{1}{y_1 - y_2} [T_\theta^{q-1}(T_\theta y_2 + (T_\theta y_1 - T_\theta y_2)z_1) - y_2]$$

Wherein

$$\bar{f}(z) = f_2(f_1(z)). \quad (2)$$

In [1], it was proved

$$\left\| f_2(z_1) - \frac{Mz_1}{1 + z_1(M - 1)} \right\|_{C^2([0,1])} \leq const \lambda^{n\epsilon} \quad (3)$$

where

$$\ln M = \sum_{i=1}^{q-1} \int_{\Delta_i} \frac{f''(y)}{2f'(y)} dy = \left(\sum_{i=0}^{q-1} \int_{\Delta_i} \frac{f''(y)}{2f'(y)} dy \right) - \int_{\Delta_0} \frac{f''(y)}{2f'(y)} dy = \ln c - \int_{\Delta_0} \frac{f''(y)}{2f'(y)} dy \tag{4}$$

Insofar as $\left| \int_{\Delta_0} \frac{f''(y)}{2f'(y)} dy \right| \leq \text{const } \lambda^n$, we get

$$\left\| f_2(z_1) - \frac{cz_1}{1+z_1(c-1)} \right\|_{C^2([0,1])} \leq \text{const } \lambda^{n\varepsilon} \tag{5}$$

It is easy to see that function $f_1(z_0)$ is close to piecewise linear function $f_d(z_0)$, where

$$f_d(z_0) = \begin{cases} \frac{z_0}{c^2(1-d)+d}, & z_0 \in [0, d] \\ \frac{d(1-c^2)+z_0c^2}{c^2(1-d)+d}, & z_0 \in [d, 1] \end{cases} \tag{6}$$

Since $|\Delta_0| \leq \text{const } \lambda^n$ is valid estimate:

Using (2)-(6) we obtain (1).

Theorem 1 implies that $\bar{f}(z)$ is convex at $0 < c < 1$ and concave at $c > 1$. Indeed, by direct calculation it is easy to verify that

$$\begin{aligned} \frac{d^2}{dz^2} F_d(z) &\geq 2c^2(1-c), \quad z \neq d \quad \text{при } 0 < c < 1 \\ \frac{d^2}{dz^2} F_d(z) &\leq -\frac{2}{c^3}(c-1), \quad z \neq d \quad \text{при } c > 1. \end{aligned}$$

Let's put

$$N = \left[\frac{1}{\varepsilon \ln \lambda} \ln\left(\frac{1}{c^3} |c-1| \min\left(\frac{1}{c^3}, c^2\right)\right) \right].$$

Denote the interval $I\left(\frac{p}{q}\right) = \left[\Omega_1\left(\frac{p}{q}\right), \Omega_2\left(\frac{p}{q}\right) \right]$.

Let's put $J = [0, 1] \setminus \bigcup_{0 \leq \frac{p}{q} \leq 1} I\left(\frac{p}{q}\right)$. Denote the Lebesgue measure on $[0, 1]$ by λ .

We now formulate the main results of our work.

Theorem 2. For all $n > N$, the following statements are true:

(a) if $\Omega = \Omega_1\left(\frac{p}{q}\right)$ or $\Omega = \Omega_2\left(\frac{p}{q}\right)$, then T_Ω has a unique periodic trajectory of period q ;

(c) at $\Omega \in \left(\Omega_1\left(\frac{p}{q}\right), \Omega_2\left(\frac{p}{q}\right)\right)$ there are equal to two periodic trajectories of period q .

Theorem 3. The Lebesgue measure of the set J is equal to zero, i.e. $\lambda(J) = 0$.

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